

**Intermediate Financial Theory**  
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**Exercises**

## Chapter 1

- 1.1. Let  $U(\cdot)$  be a well-behaved utility function that represents the preferences of an agent. Let  $f(U)$  be a monotone transformation of the original utility function  $U$ . Why is an increasing function  $g = f(U(\cdot))$  also a utility function representing the same preferences as  $U(\cdot)$ ?
- 1.2. Assume a well-behaved utility function. The maximizing choice for a consumer is preserved under increasing monotone transformations. Show this using the (first-order) optimality condition (MRS = price ratio) for a typical consumer and give an economic interpretation.
- 1.3. The MRS (marginal rate of substitution) is not constant in general. Give an economic interpretation. When will the MRS be constant? Give an example (a utility function over two goods) and compute the MRS. Is this of interest for us? Why? What undesirable properties does this particular utility function (and/or the underlying preferences) exhibit? In a two goods/two agents setting, what about the Pareto set when indifference curves are linear for both agents?
- 1.4. Consider a two goods/two agents pure exchange economy where agents' utility functions are of the form:

$$U(c_1^j, c_2^j) = (c_1^j)^\alpha (c_2^j)^{1-\alpha}, \quad j = 1, 2$$

- with  $\alpha = 0.5$ . Initial endowments are  $e_1^1 = 6, e_2^1 = 4, e_1^2 = 14, e_2^2 = 16$  (superscripts represent agents).
- a. Compute the original utility level for both agents. Compute the original MRS and give the (first-order) optimality conditions. What is your conclusion?
  - b. Describe the Pareto set (the set of Pareto optima).

c. Assume that there exists a competitive market for each good. What is the equilibrium allocation? What are equilibrium prices? Comment. What are the utility levels and MRS after trading? What do you conclude?

d. Assume that the utility functions are now given by

$$U(c_1^j, c_2^j) = \ln\left((c_1^j)^\alpha \cdot (c_2^j)^{1-\alpha}\right), \quad \alpha \in [0,1], \quad j = 1, 2.$$

What is the optimality relation for the typical consumer? How does it compare with that obtained at point  $a$ ? Compute the original utility levels and MRS. What can you say about them as compared to those obtained in item  $a$ ?

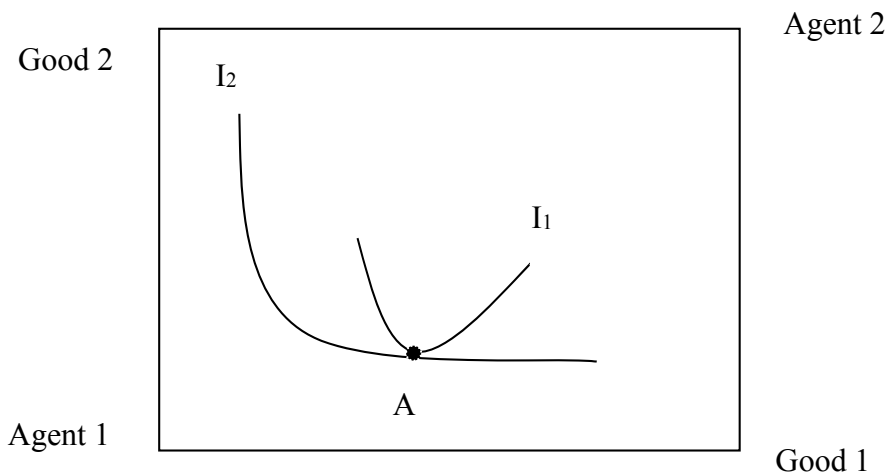
e. Same setting as in point  $d$ : What is the equilibrium allocation if agents can trade the goods on competitive markets? What are the equilibrium prices? What are the after-trade utility levels and MRS? How do they compare with those obtained under  $a$ ? What do you conclude?

1.5. Figure E.1 shows an initial endowment point  $W$ , the budget line, and the optimal choices for two agents. In what direction will the budget line move? Why?

1.6. Assume a 2 goods-2 agents economy and well-behaved utility functions. Explain why the competitive equilibrium should be on the contract curve.

1.7. What is unusual in Figure 1 below? Is there a PO allocation? Can it be obtained as a competitive equilibrium? What is the corresponding assumption of the 2<sup>nd</sup> theorem of welfare, and why is it important?

Figure 1.2 : The Edgeworth-Bowley Box: An Unusual Configuration



### Chapter 3

3.1. Utility function: Under certainty, any increasing monotone transformation of a utility function is also a utility function representing the same preferences. Under uncertainty, we must restrict this statement to linear transformations if we are to keep the same preference representation. Give a mathematical as well as an economic interpretation for this.

Check it with this example. Assume an initial utility function attributes the following values to three perspectives:

$$B \quad u(B) = 100$$

$$M \quad u(M) = 10$$

$$P \quad u(P) = 50$$

a. Check that with this initial utility function, the lottery  $L = (B, M, 0.50) \succ P$ .

b. The proposed transformations are  $f(x) = a + bx$ ,  $a \geq 0$ ,  $b > 0$  and  $g(x) = \ln(x)$ . Check that under  $f$ ,  $L \succ P$ , but that under  $g$ ,  $P \succ L$ .

3.2. Lotteries: Discuss the equivalence between  $(x, z, \pi)$  and  $(x, y, \pi + (1 - \pi)\tau)$  when  $z = (x, y, \tau)$ . Can you think of circumstances under which they would not be viewed as equal?

3.3. Intertemporal consumption: Consider a two-date (one-period) economy and an agent with utility function over consumption:

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

at each period. Define the intertemporal utility function as  $V(c_1, c_2) = U(c_1) + U(c_2)$ . Show (try it mathematically) that the agent will always prefer a smooth consumption stream to a more variable one with the same mean, that is,

$$U(\bar{c}) + U(\bar{c}) > U(c_1) + U(c_2)$$

if  $\bar{c} = \frac{c_1 + c_2}{2}$

## Chapter 4

4.1. Risk aversion: Consider the following utility functions (defined over wealth  $Y$ ):

(1)  $U(Y) = -\frac{1}{Y}$

(2)  $U(Y) = \ln Y$

(3)  $U(Y) = -Y^\gamma$

(4)  $U(Y) = -\exp(-\gamma Y)$

(5)  $U(Y) = \frac{Y^\gamma}{\gamma}$

(6)  $U(Y) = \alpha Y - \beta Y^2$

- a. Check that they are well behaved ( $U' > 0$ ,  $U'' < 0$ ) or state restrictions on the parameters so that they are [utility functions (1) – (6)]. For utility function (6), take positive  $\alpha$  and  $\beta$ , and give the range of wealth over which the utility function is well behaved.
  - b. Compute the absolute and relative risk-aversion coefficients.
  - c. What is the effect of the parameter  $\gamma$  (when relevant)?
  - d. Classify the functions as increasing/decreasing risk-aversion utility functions (both absolute and relative).
- 4.2. Certainty equivalent:

$$(1) U = -\frac{1}{Y}$$

$$(2) U = \ln Y$$

$$(3) U = \frac{Y^\gamma}{\gamma}$$

Consider the lottery  $L_1 = (50,000;10,000;0.50)$ . Determine the lottery  $L_2 = (x;0;1)$  that makes an agent indifferent to lottery  $L_1$  with utility functions (1), (2), and (3) as defined.

For utility function (3), use  $\gamma = \{0.25,0.75\}$ . What is the effect of changing the value of  $\gamma$ ?

Comment on your results using the notions of risk aversion and certainty equivalent.

- 4.3. Risk premium: A businesswoman runs a firm worth CHF 100,000. She faces some risk of having a fire that would reduce her net worth according to the following three states,  $i = 1, 2, 3$ , each with probability  $\pi(i)$  (Scenario A).

State	$\pi(i)$	Worth
1	0.01	1
2	0.04	50,000
3	0.95	100,000

Of course, in state 3, nothing detrimental happens, and her business retains its value of CHF 100,000.

- a. What is the maximum amount she will pay for insurance if she has a logarithmic utility function over final wealth? (*Note:* The insurance pays CHF 99,999 in the first case; CHF 50,000 in the second; and nothing in the third.)
- b. Do the calculations with the following alternative probabilities:

	Scenario B	Scenario C
$\pi(1)$	0.01	0.02
$\pi(2)$	0.05	0.04
$\pi(3)$	0.94	0.94

Is the outcome (the comparative change in the premium) a surprise? Why?

- 4.4. Consider two investments  $A$  and  $B$ . Suppose that their returns,  $\tilde{r}_A$  and  $\tilde{r}_B$ , are such that  $\tilde{r}_A = \tilde{r}_B + \mathcal{G}$ , where  $\mathcal{G}$  is a nonnegative random variable. Show that  $A$  FSD  $B$ .
- 4.5. Four-part question:
- a. Explain intuitively the concept of first-order stochastic dominance.

- b. Explain intuitively the concept of second-order stochastic dominance.
- c. Explain intuitively the mean variance criterion.
- d. You are offered the following two investment opportunities.

<i>Investment A</i>		<i>Investment B</i>	
<b>Payoff</b>	<b>Probability</b>	<b>Payoff</b>	<b>Probability</b>
2	0.25	1	0.333
4	0.5	6	0.333
9	0.25	8	0.333

apply concepts *a–d*. Illustrate the comparison with a graph.

- 4.6. An individual (operating in perfect capital markets) with a zero initial wealth, and the utility function  $U(Y) = Y^{\frac{1}{2}}$  is confronted with the gamble  $(16, 4; \frac{1}{2})$ .
- a. What is his certainty equivalent for this gamble?
  - b. If there was an insurance policy that, together with the original gamble, would guarantee him the expected payoff of the gamble, what is the maximum premium he would be willing to pay for it?
  - c. What is the minimum required increase (the probability premium) in the probability of the high-payoff state so that he will not be willing to pay any premium for such an insurance policy? (Note that the insurance policy still pays the expected payoff of the unmodified gamble)
  - d. Now assume that he is confronted with the gamble  $(36, 16; \frac{1}{2})$ . Calculate the certainty equivalent, the insurance premium, and the probability premium for this case as well. Explain what is going on, and why?



4.7. Refer to Table 4.2. Suppose the return data for investment 3 was as follows. Is it still the case

<i>Investment 3</i>	
<b>Payoff</b>	<b>Probability</b>
3	0.25
4	0.5
12	0.25

that investment 3 SSD investment 4?

4.8. Consider two investments with the following characteristics:

		<i>States</i>		
		$\theta_1$	$\theta_2$	$\theta_3$
	$\pi$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
<i>Returns</i>	$\tilde{z}$	10	0	10
	$\tilde{y}$	0	10	20

a. Is there state-by-state dominance between these two investments?

b. Is there FSD between these two investments?

4.9. If you are exposed to a 50/50 probability of gaining or losing CHF 1'000 and an insurance that removes the risk costs CHF 500, at what level of wealth will you be indifferent between taking the gamble or paying the insurance? That is, what is your certainty equivalent wealth for this gamble? Assume that your utility function is  $U(Y) = -1/Y$ . What would the solution be if the utility function were logarithmic?

4.10. Assume that you have a logarithmic utility function on wealth  $U(Y)=\ln Y$  and that you are faced with a 50/50 probability of winning or losing CHF 1'000. How much will you pay to avoid this risk if your current level of wealth is CHF 10'000? How much would you pay if your level of wealth is CHF 1'000'000? Did you expect that the premium you were willing to pay would increase/decrease? Why?

4.11. Assume that security returns are normally distributed. Compare portfolios A and B, using both first and second-order stochastic dominance :

Case 1	Case 2	Case 3
$\sigma_a > \sigma_b$	$\sigma_a = \sigma_b$	$\sigma_a < \sigma_b$
$E_a = E_b$	$E_a > E_b$	$E_a < E_b$

4.12. An agent faces a risky situation in which he can lose some amount of money with probabilities given in the following table:

Loss	Probability
1000	10%
2000	20%
3000	35%
5000	20%
6000	15%

This agent has a utility function of wealth of the form

$$U(Y) = \frac{Y^{1-\gamma}}{1-\gamma} + 2$$

His initial wealth level is 10000 and his  $\gamma$  is equal to 1.2.

a. Calculate the certainty equivalent of this prospect for this agent. Calculate the risk premium.

What would be the certainty equivalent of this agent if he would be risk neutral?

b. Describe the risk premium of an agent whose utility function of wealth has the form implied by the following properties:  $U'(Y) > 0$  and  $U''(Y) < 0$

4.13. An agent with a logarithmic utility function of wealth tries to maximize his expected utility. He faces a situation in which he will incur a loss of L with probability  $\pi$ . He has the possibility to insure against this loss. The insurance premium depends on the extent of the coverage. The amount covered is denoted by h and the price of the insurance per unit of coverage is p (hence the amount he has to spend on the insurance will be hp).

a. Calculate the amount of coverage h demanded by agent as a function of his wealth level Y, the loss L, the probability  $\pi$  and the price of the insurance p.

b. What is the expected gain of an insurance company offering such a contract ?

c. If there is perfect competition in the insurance market ( zero profit), what price p will the insurance company set?

d. What amount of insurance will the agent buy at the price calculated under c. What is the influence of the form of the utility function ?

4.14. Given the following probability distributions for risky payoffs  $\tilde{x}$  and  $\tilde{z}$  :

x	Probability (x)	z	Probability (z)
-10	.1	2	.2
5	.4	3	.5
10	.3	4	.2
12	.2	30	.1

a. If the only available choice is 100 % of your wealth in  $\tilde{x}$  or 100 % in  $\tilde{z}$  and you choose on the basis of mean and variance, which asset is preferred ?

b. According to the second-order stochastic dominance criterion, how would you compare them ?

4.15. There is an individual with a well-behaved utility function, and initial wealth Y. Let a lottery offer a payoff of G with probability  $\pi$  and a payoff of B with probability  $1-\pi$ .

a. If the individual already owns this lottery denote the minimum price he would sell it for by  $P_s$ . Write down the expression  $P_s$  has to satisfy.

b. If he does not own it, write down the expression  $P_b$  (the maximum price he would be willing to pay for it) has to satisfy.

c. Assume now that  $\pi=1/2$ ,  $Y=10$ ,  $G=6$ ,  $B=26$ , and the utility function is  $U(Y)=Y^{1/2}$ . Find buying and selling prices. Are they equal? Explain why not. Generally, can they ever be equal?

4.16. Consider the following investments:

Investment 1		Investment 2	
Payoff	Prob.	Payoff	Prob.
1	0.25	3	0.33
7	0.50	5	0.33
12	0.25	8	0.34

Check that neither investment dominates the other on the basis of

- The Mean-Variance criterion
- First Order Stochastic Dominance
- Second Order Stochastic Dominance

How could you rank these investments ?

## Chapter 5

- 5.1. Consider the portfolio choice problem of a risk-averse individual with a strictly increasing utility function. There is a single risky asset and a risk-free asset. Formulate an investor's choice problem and comment on the first-order conditions. What is the minimum risk premium required to induce the individual to invest all his wealth in the risky asset? (Find your answer in terms of his initial wealth, absolute risk-aversion coefficient, and other relevant parameters.) *Hint:* Take a Taylor series expansion of the utility of next period's random wealth.
- 5.2. Portfolio choice (with expected utility): An agent has  $Y = 1$  to invest. On the market two financial assets exist. The first one is riskless. Its price is one and its return is 2. Short selling on this asset is allowed. The second asset is risky. Its price is 1 and its return  $\tilde{z}$ , where  $\tilde{z}$  is a random variable with probability distribution:

$$z = 1 \text{ with probability } p_1$$

$$z = 2 \text{ with probability } p_2$$

$$z = 3 \text{ with probability } p_3$$

No short selling is allowed on this asset.

- a. If the agent invests  $a$  in the risky asset, what is the probability distribution of the agent's portfolio return ( $\tilde{R}$ )?
- b. The agent maximizes a von Neumann-Morgenstern utility ( $U$ ). Show that the optimal choice of  $a$  is positive if and only if the expectation of  $\tilde{z}$  is greater than 2.  
*Hint:* Find the first derivative of  $U$  and calculate its value when  $a = 0$ .
- c. Give the first-order condition of the agent's problem.

- d. Find  $a$  when  $U(Y) = 1 - \exp(-bY)$ ,  $b > 0$  and when  $U(Y) = \left(\frac{1}{1-\gamma}\right) Y^{1-\gamma}$ ,  $0 < \gamma < 1$ . If  $Y$  increases, how will the agent react?
- e. Find the absolute risk aversion coefficient ( $R_A$ ) in either case.
- 5.3. Risk aversion and portfolio choice: consider an economy with two types of financial assets—one risk-free and one risky asset. The rate of return offered by the risk-free asset is  $r_f$ . The rate of return of the risky asset is  $\tilde{r}$ . Note that the expected rate of return  $E(\tilde{r}) > r_f$ .

Agents are risk averse. Let  $Y_0$  be the initial wealth. The purpose of this exercise is to determine the optimal amount  $a$  to be invested in the risky asset as a function of the *Absolute Risk Aversion Coefficient* (Theorem 5.4).

The objective of the agents is to maximize the expected utility of terminal wealth:

$$\max_a E(U(Y))$$

where:

$E$  is the expectation operator,

$U(\cdot)$  is the utility function with  $U' > 0$  and  $U'' < 0$ ,

$Y$  is the wealth at the end of the period,

$a$  is the amount being invested in the risky asset.

- a. Determine the final wealth as a function of  $a$ ,  $r_f$ , and  $\tilde{r}$ .
- b. Compute the FOC. Is this a maximum or a minimum?
- c. We are interested in determining the sign of  $\frac{da^*}{dY_0}$ .

Calculate first the total differential of the FOC as a function of  $a$  and  $Y_0$ . Write the expression for  $\frac{da^*}{dY_0}$ . Show that the sign of this expression depends on the sign of its numerator.

- d. You know that  $R_A$ , the absolute risk aversion coefficient, is equal to  $\frac{-U''(\cdot)}{U'(\cdot)}$ . What does it mean if  $R'_A = \frac{dR_A}{dY} < 0$ ?
- e. Assuming  $R'_A < 0$ , compare  $R_A(Y)$  and  $R_A(Y_0(1 + r_f))$ : is  $R_A(Y) > R_A(Y_0(1 + r_f))$  or vice versa? Don't forget there are two possible cases: 
$$\begin{cases} \tilde{r} \geq r_f \\ \tilde{r} < r_f \end{cases}$$
- f. Show that  $U''(Y_0(1 + r_f) + a(\tilde{r} - r_f))(\tilde{r} - r_f) > -R_A(Y_0(1 + r_f)) \times U'(Y_0(1 + r_f)) + a(\tilde{r} - r_f)(\tilde{r} - r_f)$  for both cases in point e.
- g. Finally, compute the expectation of  $U''(Y)(\tilde{r} - r_f)$ . Using the FOC, determine its sign. What can you conclude about the sign of  $\frac{da^*}{dY_0}$ ? What was the key assumption for the demonstration?
- 5.4. Suppose that a risk-averse individual can only invest in two risky securities A and B, whose future returns are described by identical but independent probability distributions. How should he allocate his given initial wealth (normalized to 1 for simplicity) among these two assets so as to maximize the expected utility of next period's wealth?
- 5.5. An individual with a well-behaved utility function and an initial wealth of \$1 can invest in two assets. Each asset has a price of \$1. The first is a riskless asset that pays \$1. The second pays amounts  $a$  and  $b$  (where  $a < b$ ) with probabilities of  $\pi$  and  $(1 - \pi)$ , respectively. Denote the units demanded of each asset by  $x_1$  and  $x_2$ , respectively, with  $x_1, x_2 \in [0, 1]$ .

- a. Give a simple necessary condition (involving  $a$  and  $b$  only) for the demand for the riskless asset to be strictly positive. Give a simple necessary condition (involving  $a$ ,  $b$ , and  $\pi$  only) so that the demand for the risky asset is strictly positive.
- b. Assume now that the conditions in item (a) are satisfied. Formulate the optimization problem and write down the FOC. Can you intuitively guess the sign of  $dx_1/da$ ? Verify your guess by assuming that  $x_1$  is a function of  $a$  written as  $x_1(a)$ , and taking the total differential of the FOC with respect to  $a$ . Can you also conjecture a sign for  $dx_1/d\pi$ ? Provide an economic interpretation without verifying it as done previously.

5.6. An individual with a utility function  $U(c) = -\exp(-Ac)$  (where  $A=(1/30)\ln 4$ ) and an initial wealth of \$50 must choose a portfolio of two assets. Each asset has a price of \$50. The first asset is riskless and pays off \$50 next period in each of the two possible states. The risky asset pays off  $z_s$  in state  $s=1,2$ . Suppose also that the individual cares only about next period consumption (denoted by  $c_1$  or  $c_2$  depending on the state). The probability of state 1 is denoted by  $\pi$ .

a. If the individual splits his wealth equally between the two assets, fill in the following table assuming that each of three scenarios is considered.

Scenario	$(z_1, z_2)$	$\pi$	$(c_1, c_2)$	$E(c)$	$\text{Var}(c)$
1	(20, 80)	1/5			
2	(38, 98)	1/2			
3	(30, 90)	1/3			

b. How would this individual rank these three scenarios? Explain and give reasons to support your argument.

c. Show that under each scenario the individual's optimal decision is to invest an equal amount on each of the two assets.

5.7. Consider an agent with a well-behaved utility function who must balance his portfolio between a riskless asset and a risky asset. The first asset, with price  $p_1$  has the certain payoff  $z_1$  while the second asset, with price  $p_2$  pays off  $\tilde{z}_2$  which is a random variable. The agent has an initial wealth  $Y_0$  and he only cares about next period. Assuming that he holds  $x_1$  units of the riskless asset and  $x_2$  units of the risky asset;

a. Write down his expected utility function in terms of  $Y_0$ ,  $x_1$ ,  $x_2$ ,  $z_1$ , and  $\tilde{z}_2$ . Write down his budget constraint as well. Find the equation (the FOC) that the optimal demand for the risky asset has to satisfy.

b. Assume now that his utility function is of the form:  $U(Y)=a-be^{-AY}$  with  $a, b>0$ . Show that the demand for the risky asset is independent of the initial wealth. Explain intuitively why this is so.

5.8 Consider the savings problem of Section 4.4 :

$$\max_{s \geq 0} EU\{(y_0 - s) + \delta U(s\tilde{x})\}$$

$$\text{Assume } U(c) = Ec - \frac{1}{2} \chi \sigma_c^2$$

Show that if  $\tilde{x}_A$  SSD  $\tilde{x}_B$  ( $w/E\tilde{x}_A = E\tilde{x}_B$ ), then  $s_A > s_B$ .

## Chapter 6

6.1. Consider an equally weighted portfolio of three stocks, each of which is independently distributed of the others [that is,  $cov(r_i, r_j) = 0$  for different securities  $i$  and  $j$ ]. Assume also that each stock has the same total risk ( $\sigma$ ). What fraction of each stock's risk is diversified away by including it in this portfolio?

6.2. At the moment, all of your assets are invested in asset A with the following return and risk characteristics:

$$E\tilde{r}_A = 10\%$$

$$\sigma_A = 10\%$$

Another asset (call it "B") becomes available; the characteristics of B are as follows;

$$E\tilde{r}_B = 20\%, \sigma_B = 25\%. \text{ Furthermore, the correlation of A's and B's return patterns is } -1.$$

By reallocating your portfolio to include some of asset B, how much additional return could you expect to receive if you wanted to maintain your portfolio's risk at  $\sigma = 10\%$ . *Hint: Solve for  $w_B$ , not for  $w_A$ .*



- 6.3. You are a portfolio manager considering whether or not to allocate some of the money with which you are entrusted to the market index of Australian stocks. Your assistant provides you with the following historical return information:

<i>Returns</i>		
<b>Your Portfolio</b>	<b>Australian Index</b>	
1992	54%	50%
1993	24%	10%
1994	6%	10%
1995	24%	60%
1996	6%	20%
1997	54%	80%

- a. Show that the addition of the Australian index (AUS) to your portfolio (your) will reduce risk (at no loss in returns) provided

$$\text{corr}(\text{your}, \text{AUS}) < \frac{\sigma_{\text{your}}}{\sigma_{\text{AUS}}}$$

(assuming, as in the case,  $\sigma_{\text{AUS}} > \sigma_{\text{your}}$ )

- b. Based on this historical data could you receive higher returns for the same level of risk (standard deviation) by allocating some of your wealth to the Australian index?
- c. Based on historical experience, would it be possible to reduce your portfolio's risk below its current level by investing something in the Australian index?
- d. What fraction of the variation in Australian stocks can be explained by variation in your portfolio's returns?

## Chapter 8

- 8.1. Comment fully on the following statement: If a portfolio has a high  $\beta$ , then further diversification is possible.
- 8.2. What is the difference between the relationship implied by the Capital Market Line (CML) and the Security Market Line (SML)? Consider a particular portfolio  $P$  with risk  $\sigma_P$ . Under what circumstances will the CML and the SML give the same  $E\tilde{r}_P$ ?
- 8.3. Among your numerous assets, you are the owner of a finance company that extends one-year loans to people to buy appliances and other household goods. A young finance whiz that you just hired suggests that since the default risk of your loans is entirely diversifiable, you should charge your customers (those who are borrowing from you) the risk-free rate.
- What do you think of the suggestion?
  - Assume that the risk-free rate is 10 percent and that the probability of default is 5 percent for the next year on a typical loan. In addition, assume that if a borrower defaults, all the principal but no interest is repaid. What rate should your finance company charge for loans over the next year?
  - Suppose that the reclaimed appliances have lost 20 percent of their original value and that in the event of default no interest is paid. What rate should you set?
- 8.4. In the CAPM setting, it is argued that only a fraction of the total risk of a particular asset is priced. Use the CML and SML to prove this assertion.
- 8.5. Consider two fully isolated economies, economy 1 and economy 2. The same assets are traded in both economies, but the average investor in economy 2 is more risk averse than the average investor in economy 1. Compare the CMLs in both economies.

- 8.6. Consider two fully isolated economies. Asset returns in economy 2 are, in general, more positively correlated than asset returns in economy 1. Compare the CMLs in both economies.
- 8.7. Under the CAPM, all investors form portfolios of two assets, a risk-free asset and a risky portfolio  $M$ , regardless of their level of wealth  $Y$ . If an investor becomes wealthier, he may want to increase or decrease the proportion of his wealth held in the risky portfolio, and, by the Cass–Stiglitz Theorem, unless his preferences have a very specific form, as his wealth changes, he will want to alter the composition of the risky part of his portfolio. The CAPM does not assume such preference restrictions. Yet, the CAPM equilibrium does not seem to permit the desired changes in the composition of the agent’s risky portfolio! Is there a contradiction?
- 8.8. Consider the following three assets:

$$e = \begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ and } V = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 9 \end{pmatrix}$$

Compute  $g$  and  $h$  from our portfolio composition characterization. Identify the MVP. Identify the zero covariance portfolio for asset 3.

Note that  $V^{-1} = \begin{pmatrix} A & B & C \\ B & B & C \\ C & C & D \end{pmatrix}$  where

$$A = 1.346154$$

$$B = 0.346154$$

$$C = 0.038462$$

$$D = 0.115385$$

8.9. Consider a two-period economy with  $I$  agents,  $J$  risky assets, and one risk-free rate asset.

You can write the agent's wealth as follows:

$$\tilde{Y}_i = \left( Y_0^i - \sum_j x_j^i \right) (1 + r_f) + \sum_j x_j^i (1 + \tilde{r}_j)$$

where

$Y_0^i$  = initial wealth of agent  $i$ ,

$r_f$  = the risk-free interest rate,

$\tilde{r}_j$  = the random rate of return on the  $j$ th risky asset,

$x_j^i$  = the amount invested in the  $j$ th asset by agent  $i$ .

As usual, the individual's choice problem is:

$$\max_{x_j^i} E [U_i(\tilde{Y}_i)]$$

a. Show that the FOC can be written as follows:

$$E [U_i'(\tilde{Y}_i)(\tilde{r}_j - r_f)] = 0 \quad (1)$$

b. Show that Equation (1) can be written as follows:

$$E [U_i'(\tilde{Y}_i)] E(\tilde{r}_j - r_f) = -Cov [U_i'(\tilde{Y}_i), \tilde{r}_j] \quad (2)$$

Recall that  $Cov(x, y) = E(xy) - E(x)E(y)$ .

c. Using the following property of the covariance  $Cov[g(x), y] = E[g'(x)] Cov(x, y)$ , show that

Equation (2) can be rewritten as follows:

$$E [U_i'(\tilde{Y}_i)] E(\tilde{r}_j - r_f) = -E [U_i''(\tilde{Y}_i)] Cov(\tilde{Y}_i, \tilde{r}_j) \quad (3)$$

d. Aggregating on all individuals (summing on  $i$ ), show that Equation (3) can be rewritten as follows:

$$E(\tilde{r}_j - r_f) = \frac{Y_{M0}}{\sum_i \frac{1}{R_{Ai}}} \text{Cov}(\tilde{r}_M, \tilde{r}_j) \quad (4)$$

where  $R_{Ai} = \frac{-E[U'_i(\tilde{Y}_i)]}{E[U''_i(\tilde{Y}_i)]}$  is reminiscent of the absolute risk aversion coefficient and

$$\sum_i \tilde{Y}_i = Y_{M0}(1 + \tilde{r}_M).$$

e. Show that Equation (4) implies:

$$E(\tilde{r}_M - rf) = \frac{Y_{M0}}{\sum_i \frac{1}{R_{Ai}}} \text{Var}(\tilde{r}_M) \quad (5)$$

f. Derive the traditional CAPM relationship. *Hint:* Combine Equations (4) and (5).

8.10. Show that maximizing the Sharpe ratio,  $\frac{E(r_p) - r_f}{\sigma_p}$  yields the same tangency portfolio that was obtained in the text.

*Hint:* Formulate the Lagrangian and solve the problem.

8.11. Think of a typical investor selecting his preferred portfolio along the Capital Market Line. Imagine:

1. A 1% increase in both the risk free rate and the expected rate of return on the market, so that the CML shifts in a parallel fashion
2. An increase in the expected rate of return on the market without any change in the risk free rate, so that the CML tilts upward.

In these two situations, describe how the optimal portfolio of the typical investor is modified.

8.12. Questions about the Markowitz model and the CAPM.

- a. Explain why the efficient frontier must be concave.
- b. Suppose that there are N risky assets in an economy, each being the single claim to a different firm (hence, there are N firms). Then suppose that some firms go bankrupt, i.e. their single stock disappears; how is the efficient frontier altered?
- c. How is the efficient frontier altered if the borrowing (risk-free) rate is higher than the lending rate? Draw a picture.

- d. Suppose you believe that the CAPM holds and you notice that an asset (call it asset A) is above the Security Market Line. How can you take advantage of this situation ? What will happen to stock A in the long run?
- 8.13. Consider the case without a riskless asset. Take any portfolio  $p$ . Show that the covariance vector of individual asset returns with portfolio  $p$  is linear in the vector of mean returns if and only if  $p$  is a frontier portfolio.
- Hint: To show the "if" part is straightforward. To show the converse begin by assuming that  $Vw=ae+b\mathbf{1}$  where  $V$  is the variance-covariance matrix of returns,  $e$  is the vector of mean returns, and  $\mathbf{1}$  is the vector of ones.
- 8.14. Show that the covariance of the return on the minimum variance portfolio and that on any portfolio (not only those on the frontier) is always equal to the variance of the rate of return on the MVP.
- Hint: consider a 2-assets portfolio made of an arbitrary portfolio  $p$  and the MVP, with weights  $a$  and  $1-a$ . Show that  $a=0$  satisfies the variance minimizing program; the conclusion follows.
- 8.15. Find the frontier portfolio that has an identical variance as that of its zero-covariance portfolio. (That is, determine its weights.)
- 8.16. Let there be two risky securities,  $a$  and  $b$ . Security  $a$  has expected return of 13% and volatility of 30%. Security  $b$  has expected return of 26% and volatility of 60%. The two securities are uncorrelated.
- a. Compute the portfolio on the efficient frontier that is tangent to a line from zero, the zero beta portfolio associated with that portfolio, and the minimum-variance portfolio.
- b. Assume a risk-free rate of 5%. Compute the portfolio of risky assets that investors hold. Does this portfolio differ from the tangency portfolio computed under a) ? If yes, why?
- 8.17 a. Given risk-free borrowing and lending, efficient portfolios have no unsystematic risk. True or false?
- b. If the agents in the economy have different utility functions the market portfolio is not efficient. True or false?
- c. The CAPM makes no provision for investor preference for skewness. True or false?

## Chapter 9

- 9.1. General equilibrium and uncertainty: Consider a two-period exchange economy with two agents. They have identical utility functions:

$$U(c_1, c_2(\theta)) = \ln c_1 + \ln c_2(\theta)$$

where:

$c_1$  is the consumption level at date 1

$c_2(\theta)$  is the consumption level at date 2 if state  $\theta$  occurs.

Let us assume two possible states of nature at date 2, and the following endowment structure:

<i>Agent</i>	<i>Date 1</i>	<i>Date 2</i>	
		$\theta = 1$	$\theta = 2$
I	4	1	3
II	4	3	1

- Describe *intuitively* a Pareto optimal allocation of resources. Is it unique?
- Suppose only one type of security may be issued. Is it possible to achieve the Pareto optimal allocation? If so, how?
- Suppose the commodity in the previously described economy can be *costlessly* stored. No other asset is traded. Describe intuitively how this opportunity will be exploited and why. Is the utility level of the two agents increased?
- Let us now assume that agent 1 is risk neutral and there is aggregate uncertainty; that is, at date 2 the endowment structure is

$\theta = 1$	$\theta = 2$
1	3
2	4

Discuss intuitively what the new Pareto optimal allocation should look like. Assume the existence of Arrow–Debreu securities and characterize the resulting market equilibrium.

## 9.2. Two-part problem:

- a. Think of an economy with two agents with utility functions of the form:

$$U(c) = c_0 + E(c_1(\theta)) - 2 \text{ var}(c_1(\theta))$$

where:

$c_0$  is the consumption level at date 0,

$c_1(\theta)$  is the consumption level at date 1 if state  $\theta$  occurs.

Their endowments are given by

	<i>Date t = 0</i>		<i>Date t = 1</i>	
		<b>State 1</b>		<b>State 2</b>
<i>Agent 1</i>	4	1		5
<i>Agent 2</i>	4	5		1

The state probabilities are  $\frac{1}{4}$  for state 1 and  $\frac{3}{4}$  for state 2.

1. Intuitively, what condition should a Pareto optimal allocation satisfy in this particular setup?
2. Are there several Pareto optima? If so, characterize the set of Pareto optima.
3. Is the following allocation a Pareto optimum? Why?

	<i>Date t = 0</i>		<i>Date t = 1</i>	
		<b>State 1</b>		<b>State 2</b>
<i>Agent 1</i>	4	2		2
<i>Agent 2</i>	4	4		4

- b. Now the agents have a utility function of the form:

$$U(c) = c_0 + E(\ln(c_1(\theta)))$$



where:  $c_0$  is the consumption level at date 0,  $c_1(\theta)$  is the consumption level at date 1 if state  $\theta$  occurs.

The endowments are:

	<i>Date t = 0</i>		<i>Date t = 1</i>	
		<b>State 1</b>		<b>State 2</b>
<i>Agent k = 1</i>	4	1		5
<i>Agent k = 2</i>	4	5		1

1. Describe the Pareto optimum or the set of Pareto optima for this utility function and compare it with the one under item 8.6.a.1.
2. Two Arrow–Debreu securities denoted by  $Q1$  (state 1) and  $Q2$  (state 2) are introduced. Both are in zero net supply. Compute the competitive equilibrium allocation. Is it Pareto optimal? Discuss intuitively its characteristics (the determinants of the prices of the two securities and the post-trade allocation).
3. Suppose only one Arrow–Debreu security depending on state 1 is traded. It is in zero net supply. Compute the competitive equilibrium allocation. Is it Pareto optimal?
- 9.3. Think of a simple exchange economy with two agents that have the following utility functions:

$$U_1(\cdot) = 0.25c_0 + 0.5E[\ln(c_1(\theta))]$$

$$U_2(\cdot) = c_0 + E[\ln(c_1(\theta))]$$

where  $\theta$  denotes the two equally likely states at date 1. The endowments are

	<i>Date t = 0</i>	<i>Date t = 1</i>	
		$\theta = 1$	$\theta = 2$
<i>Agent 1</i>	2	2	4
<i>Agent 2</i>	4	4	2

- a. Describe the Pareto optimum or the set of Pareto optima. Is it unique? Is perfect risk sharing achieved? Why or why not? Answer the same question if the state 2 endowments were 5 for agent 1 and 3 for agent 2.
- b. Agents can trade with two Arrow–Debreu securities denoted by Q [with payoffs (1,0)] and R [with payoffs (0,1)]. Calculate the competitive equilibrium allocation. Is it Pareto optimal? Discuss intuitively its characteristics (the determinants of the prices of the two securities and the post-trade allocation).
- c. Suppose that both securities can be traded at the beginning and there are no short-selling constraints. A firm has zero cost of introducing one unit of either security, and it wants to maximize its profits. Which security should it introduce? For which agent is this introduction more valuable? Is the outcome Pareto optimal? Why or why not?

9.4. Two-part question:

- a. To what extent should we care whether the current organization of markets leads to a Pareto optimal allocation?
- b. Are markets complete? Do we care?

## Chapter 10

10.1. CAPM versus Consumption CAPM:

- a. Contrast the two models in words.

- b. Explain why, in principle, the consumption CAPM is more satisfactory.
- c. Can you think of circumstances where the two models are essentially identical?

10.2. Consider an endowment economy identical to the one considered in the discussion of the consumption CAPM. Consider an option that entitles the owner to exercise the right to buy one unit of the asset one period in the future at a fixed price  $p^*$ . The price  $p^*$  is known at period  $t$ , whereas the option to buy is exercised at period  $t + 1$ .

Suppose the representative agent's utility function is  $U(c) = \ln(c)$ . You are asked to price the option.

Follow these steps:

- a. Write the value of the option at expiration as a function of the price of the underlying asset.
- b. Write the price of the asset at the later date as a function of the state at that date,  $\theta$ , the total quantity of good available.
- c. Given the chosen utility function, write  $q(\theta, c)$ , the Arrow–Debreu price.
- d. Use Arrow–Debreu pricing to price the option.

10.3. Consider the setting described previously with the following modification: The asset in question requires the owner to buy the asset at a price  $p^*$  (a forward contract). Price the security.

10.4. Consider the usual representative agent economy. Suppose that the representative agent has log utility,  $U(c) = \ln(c)$ . Define the wealth portfolio as a claim to all future dividends available for consumption in this economy.

- a. Show that the price of this wealth portfolio is proportional to consumption itself.

- b. Show that the return on this portfolio is proportional to consumption growth. *Hint:* Think about the definition of the return on an asset to write down the return on the wealth portfolio.
- c. Finally, express the price of a cash flow paying off \$100 at each date for the next two periods in terms of the return on the wealth portfolio.
- 10.5. Begin with the fundamental equation  $E[mR_i] = 1$  where  $R_i$  is the gross return on asset  $i$ , and  $m$  is the pricing kernel. Noting that this fundamental equation should hold for the gross returns on the riskless asset and on the market portfolio ( $R_M$ ) as well, obtain a CAPM-like expression. Define a functional relation between  $m$  and  $R_M$  so that you will obtain precisely the CAPM expression.

## Chapter 11

- 11.1. Arrow–Debreu pricing: You are given the following term structure of interest rates for three periods into the future:

$$r_1 = 0.0989, \quad r_2 = 0.1027, \quad r_3 = 0.1044$$

- a. Construct the Arrow–Debreu prices for these state-dates.
- b. Assume that there are three states in each period and that the constant Arrow–Debreu state-matrix is as follows:

$$\begin{pmatrix} 0.28 & 0.33 & 0.30 \\ 0.27 & 0.34 & 0.31 \\ 0.24 & 0.28 & 0.36 \end{pmatrix}$$

Assume we are in state 1. Are these prices consistent with the term structure just given?

- 11.2. What is the relationship between the price of an Arrow–Debreu security and the corresponding state probability? How is the price of an Arrow–Debreu security affected by a change in the discount factor? What else affects the price of an Arrow–Debreu security?

- 11.3. Arrow–Debreu pricing: Consider a world with two states of nature. The following matrix provides the one-period Arrow–Debreu prices in all situations

$t+1$	<b>State 1</b>	<b>State 2</b>
State 1	0.53	0.43
State 2	0.45	0.52

Describe the term structure of interest rates over three periods.

- 11.4. You are given the following prices of a set of coupon bonds. Construct the term structure and price of the corresponding date-contingent claim.

$t =$	0	1	2	3	4	5
bond 1	−960	1,000				
bond 2	−900	100	1,100			
bond 3	−800	120	120	1,120		
bond 4	−650	130	130	130	1,130	
bond 5	−400	150	150	150	150	1,150

- 11.5. Options and market completeness: Remember that in certain circumstances, it is not possible to achieve market completeness with call options only (why?). Show that in the following market structure with three assets and four states, introducing a put option on the first asset with exercise price 1 is sufficient to achieve market completeness (i.e., to generate a complete set of Arrow–Debreu securities for those four states).

Asset 1: (0,0,0,1)

Asset 2: (1,1,0,1)

Asset 3: (0,1,1,1)

11.6. Three-part question:

a. Describe intuitively the idea of an Arrow–Debreu security. Arrow–Debreu securities are not observed in real markets. Is the concept nevertheless useful? What is the link between Arrow–Debreu securities and options?

b. You observe the following assets with the corresponding state-dependent payoffs:

Securities	A	B	C	
States	$\begin{bmatrix} 3 & 7 & 8 \\ 1 & 2 & 9 \\ 7 & 16 & 25 \end{bmatrix}$	Is this market complete?		

c. Is the following market structure complete?

Assets	A	B	C	
States	$\begin{bmatrix} 1 & 5 & 12 \\ 2 & 3 & 6 & 14 \\ 3 & 12 & 7 & 16 \\ 4 & 15 & 8 & 20 \end{bmatrix}$			

If the market is not complete, introduce a derivative security to complete it.

Construct an Arrow–Debreu security by introducing calls or puts on these assets.

Can you reproduce the same structure by using only puts?

11.7. Consider a world with two states of nature. You have the following term structure of interest rates over two periods:

$$r_1^1 = 11.1111, \quad r_2^1 = 25.0000, \quad r_1^2 = 13.2277, \quad r_2^2 = 21.2678$$

where the subscript denotes the state at the beginning of period 1, and the superscript denotes the period. For instance  $\frac{1}{(1+r_j^2)^2}$  is the price at state  $j$  at the beginning of period 1 of a riskless asset paying 1 two periods later. Construct the stationary (same every period) Arrow-Debreu state price matrix.

11.8. A-D pricing

Consider two 5-year coupon bonds with different coupon rates which are simultaneously traded.

	<u>Price</u>	<u>Coupon</u>	<u>Maturity Value</u>
Bond 1	1300	8 %	1000
Bond 2	1200	6.5 %	1000

For simplicity, assume that interest payments are made once per year. What is the price of a 5-year A-D security when we assume the only state is the date.

11.9. You anticipate receiving the following cash flow which you would like to invest risk free.

t = 0	1	2	3
	\$1m	\$1,25m	

The period denotes one year. Risk free discount bonds of various maturities are actively traded, and the following price data is reported:

t = 0	1	2	3
-950	1000		
-880		1000	
-780			1000

- Compute the term structure implied by these bond prices.
- How much money can you create, risk free, at  $t = 3$  from the above cash flow using the three mentioned instruments?
- Show the transactions whereby you guarantee (lock in) its creation at  $t = 3$ .

11.10. Consider an exchange economy with two states. There are two agents with the same utility function  $U(c) = \ln(c)$ . State 1 has a probability of  $\pi$ . The agents are endowed with the units of the consumption good at each state. Their endowments across themselves and across states are not necessarily equal. Total endowment of this consumption good is  $e_1$  in state 1 and  $e_2$  in state 2. Arrow-Debreu state prices are denoted by  $q_1$  and  $q_2$ .

- Write down agents' optimization problems and show that

$$\frac{q_1}{q_2} = \frac{\pi}{1-\pi} \left( \frac{y_2}{y_1} \right)$$

Assuming that  $q_1 + q_2 = 1$  solve for the state prices. Hint: Recall the simple algebraic fact that  $\frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}$ .

b. Suppose there are two types of asset in the economy. A riskless asset (asset 1) pays off 1 (unit of the consumption good) in each state and has market price of  $P_1=1$ . The risky asset (asset 2) pays off 0.5 in state 1 and 2 in state 2. Aggregate supplies of the two assets are  $Q_1$  and  $Q_2$ . If the two states are equally likely, show that the price of the risky asset is

$$P_2 = \frac{5Q_1 + 4Q_2}{4Q_1 + 5Q_2}$$

Hint: Note that in this case state-contingent consumption of the agents are assured, in equilibrium, through their holdings of the two assets. To solve the problem you will need to use the results of section a). There is no need to set up another optimization problem.

## Chapter 12

- 12.1. Consider a two-date economy where there are three states of the world at date 1. Consumption per capita at date 1 will be \$5, \$10, or \$15. A security that pays one unit of consumption next period is worth \$8 today. The risk-free security pays a gross return of 1.1. A call option on per capita consumption with an exercise price of \$12 costs \$1. The probabilities of the three states are 0.3, 0.4, and 0.3, respectively.
- Are markets complete? If yes, what are the state prices?
  - Price a put option on per capita consumption with an exercise price of \$8.
  - Can you derive the risk-neutral probabilities?
  - Suppose that an investor has the following expected utility function:  $\ln c_0 + \delta E \ln c_\theta$ .
- 12.2. Arrow–Debreu pricing: Consider a two-period economy similar to the one we are used to but with the following basic data:



Agent	Period 1 Endowment	Period 2 Endowment	
		$\theta_1$	$\theta_2$
1	10	1	5
2	5	4	6

The agents have utility functions  $U(c_1, c_2(\theta)) = c_1 + E[\ln(c_2(\theta))]$ , and the state probabilities are  $\text{Prob}(\theta_1) = 1/3 = 1 - \text{Prob}(\theta_2)$ .

Construct the risk-neutral probabilities and compute the value today of agent 2's period 2 endowment.

## Chapter 13

13.1. Consider an economy where the endowment is given by the following binomial process:

The representative agent has utility function

$$u(c_0, c_1(\theta_1), c_2(\theta_2)) = \ln(c_0) + \delta E(\ln c_1) + \delta^2 E(\ln c_2),$$

and the (subjective) state probabilities are

$$\pi(\theta_{11}) = \pi(\theta_{12}) = \frac{1}{2}, \pi(\theta_{21}) = \pi(\theta_{23}) = \frac{1}{4}.$$

- Solve for the expected utility of the representative agent. Assume that  $\delta = 0.96$ .
- Solve for Arrow–Debreu prices, risk-neutral probabilities, and the pricing kernel.
- What is the value of the consumption stream at date 0?
- Price a one- and a two-period bond. Can we observe a term premia? What do you conclude?
- Price a European call on consumption at date 0. The expiration date is just immediately after the representative agent has received his endowment at date 1 (strike = 1).
- Let us assume that you are an outsider to this economy and you do not observe the representative agent. Price the call with the binomial method presented in Chapter 12. Note

that the consumption stream can be interpreted as dividends and assume that the representative agent announces the price process (!) as well as the interest rate.

- g. Do not hesitate to comment on your results, (i.e., show the unique relationship between the methods).

## Chapter 14

- 14.1. Suppose the existence of three assets with the following characteristics:

	$E(r_i)$	$b\beta_i$
A	0.07	0.5
B	0.09	1.0
C	0.17	1.5

Assume that the specific risk can be eliminated through diversification and that the return model can be written:

$$r_i = E(r_i) + b_i F_1 + e_i$$

- Plot the three assets on a  $E(r_i) - b_i$  graph.
- Using assets A and B, construct a portfolio with no systematic risk. Do the same using assets B and C.
- Construct an arbitrage portfolio and compute its expected return.
- Comment and describe how prices will adjust as a result of arbitrage opportunity.
- Suppose now that  $E(r_A) = 0.06$ ,  $E(r_B) = 0.10$ , and  $E(r_C) = 0.14$ . Plot the three assets in the graph in item a. Is it still possible to construct an arbitrage portfolio with positive expected return? Comment.

- 14.2. Assume the following market model:

$$\tilde{r}_j = a_j + \beta_{jM} \tilde{r}_M + \varepsilon_j$$

with

$\tilde{r}_j$  = the return on asset  $j$

$a_j$  = a constant

$\beta_{jM}$  = the sensitivity of asset  $j$  to fluctuations in the market return

$\tilde{r}_M$  = the market return

$\tilde{\varepsilon}_j$  = a stochastic component with

$$E(\tilde{\varepsilon}_j) = 0 \text{ and } Var(\tilde{\varepsilon}_j) = \sigma_{\varepsilon_j}^2, \text{ and } cov(\tilde{\varepsilon}_i, \varepsilon_j) = 0, \forall i, j.$$

- a. Prove that  $\sigma_j^2 = \beta_j^2 \sigma_M^2 + \sigma_{\varepsilon_j}^2$ .
- b. Prove that  $\sigma_{ij} = \beta_i \beta_j \sigma_M^2$ .

#### 14.3. CAPM and APT:

- a. Briefly describe these two models: Capital Asset Pricing Model and Arbitrage Pricing Theory? What are their main assumptions?
- b. CAPM and APT are so-called valuation models. In fact, they permit us to compute the *expected rate of return* of a security. How can this be used to derive a *current price*?
- c. Contrast the two models. Are they compatible? Are they identical under certain conditions?

#### 14.4. Draw the parallels and underline the key differences between the Arbitrage Pricing Theory (APT) and Arrow–Debreu pricing.

#### 14.5. The APT does not assume individuals have homogeneous beliefs concerning the random returns of the assets under consideration. True or false? Comment.

#### 14.6. Assume that the following two-factor model describes returns

$$r_i = a_i + b_{i1}F_1 + b_{i2}F_2 + e_i$$

Assume that the following three portfolios are observed.

Portfolio	Expected returns	$b_{i1}$	$b_{i2}$
A	12.0	1	0.5
B	13.4	3	0.2
D	12.0	3	-0.5

a. Find the equation of the plane that must describe equilibrium returns.

b. If  $\tilde{r}_M - r_f = 4$ , find the values for the following variables that would make the expected returns consistent with equilibrium determined by the CAPM.

i)  $r_f$

ii)  $\beta_{pi}$ , the market beta of the pure portfolio associated with factor i

14.7. Based on a single factor APT model, the risk premium on a portfolio with unit sensitivity is 8% ( $\lambda_1 = 8\%$ ). The risk free rate is 4%. You have uncovered three well-diversified portfolios with the following characteristics:

<u>Portfolio</u>	<u>Factor Sensitivity</u>	<u>Expected Return</u>
A	.80	10.4%
B	1.00	10.0%
C	1.20	13.6%

Which of these three portfolios is not in line with the APT?

14.8 A main lesson of the CAPM is that “diversifiable risk is not priced”. Is this important result supported by the various asset pricing theories reviewed in this book? Discuss.

*As a provision of supplementary material we describe below how the APT could be used to construct an arbitrage portfolio to profit of security mispricing. The context is that of equity portfolios. The usefulness of such an approach will, of course, depend upon “getting the right set of factors” so that the attendant regressions have high  $R^2$ .*

Supplementary discussion: An APT exercise in practice

- Step 1: select the factors; suppose there are J of them.
- Step 2: For a large number of firms N (big enough so that when combined in an approximately equally weighted portfolio of them the unique risks diversify away to approx. zero) undertake the following time series regressions on historical data:

$$\text{Firm 1: IBM} \quad \tilde{r}_{\text{IBM}} = \hat{\alpha}_{\text{IBM}} + \hat{b}_{\text{IBM},1} \tilde{F}_1 + \dots + \hat{b}_{\text{IBM},J} \tilde{F}_J + \tilde{e}_{\text{IBM}}$$

$$\begin{array}{lcl}
\text{Firm 2: BP} & \tilde{r}_{BP} = \hat{\alpha}_{BP} + \hat{b}_{BP,1}\tilde{F}_1 + \dots + \hat{b}_{BP,J}\tilde{F}_J + \tilde{\epsilon}_{BP} & \\
\vdots & \vdots & \vdots \\
\text{Firm N: GE} & \tilde{r}_{GE} = \hat{\alpha}_{GE} + \hat{b}_{GE,1}\tilde{F}_1 + \dots + \hat{b}_{GE,J}\tilde{F}_J + \tilde{\epsilon}_{GE} & 
\end{array}$$

The return to each stock are regressed on the same J factors; what differs is the factor sensitivities  $\{\hat{b}_{IBM,1}, \dots, \hat{b}_{GE,J}\}$ . Remember that:

$$\hat{b}_{BP,J} = \frac{\text{cov}(\tilde{r}_{BP}, \tilde{F}_j^{\text{HIST}})}{\sigma_{F_j}^{\text{HIST}}}$$

(want high  $R^2$ )

3. Step 3: first assemble the following data set:

$$\begin{array}{lcl}
\text{Firm 1: IBM} & AR_{IBM}, \hat{b}_{IBM,1}, \hat{b}_{IBM,2}, \dots, \hat{b}_{IBM,J} & \\
\text{Firm 2: BP} & AR_{BP}, \hat{b}_{BP,1}, \hat{b}_{BP,2}, \dots, \hat{b}_{BP,J} & \\
\vdots & \vdots & \vdots \\
\text{Firm N: GE} & AR_{GE}, \hat{b}_{GE,1}, \hat{b}_{GE,2}, \dots, \hat{b}_{GE,J} & 
\end{array}$$

The  $AR_{IBM}, AR_{BP}, \dots$  etc. represent the average returns on the N stocks over the historical period chosen for the regression.

Then, regress the average returns on the factor sensitivities (we have N data points corresponding to the N firms)

(these vary across the N firms)

$$A\tilde{r}_i = r_f + \hat{\lambda}_1 \tilde{b}_{i1} + \hat{\lambda}_2 \tilde{b}_{i2} \dots \hat{\lambda}_J \tilde{b}_{iJ}$$

we obtain estimates  $\{\hat{\lambda}_1, \dots, \hat{\lambda}_J\}$

In the regression sense this determines the “best” linear relationship among the factor sensitivities and the past average returns for this sample of N stocks.

This is a “cross sectional” regression. (Want a high  $R^2$ )

4. Step 4: Compare, for the N assets, their actually observed returns with what should have been observed given their factor sensitivities; compute  $\alpha_j$ 's :

$$\alpha_{IBM} = \underbrace{AR_{IBM}}_{\text{IBM's actually observed historical return}} - \underbrace{\left[ r_f + \hat{\lambda}_1 \hat{b}_{IBM,1} + \dots + \hat{\lambda}_J \hat{b}_{IBM,J} \right]}_{\text{predicted return given its factors intensities } \hat{b}_{IBM,1}, \dots, \hat{b}_{IBM,J} \text{ according to the regression in step 3}}$$

$$\vdots$$

$$\alpha_{GE} = AR_{GE} - \left[ r_f + \hat{\lambda}_1 \hat{b}_{GE,1} + \dots + \hat{\lambda}_J \hat{b}_{GE,J} \right]$$

Note that

$\alpha_j > 0$  implies the average returns exceeded what would be justified by the factor intensities => undervalued;

$\alpha_j < 0$  implies the average returns fell short of what would be justified by the factor intensities => overvalued.

e. Step 5: Form an arbitrage portfolio of the N stocks:

if  $\alpha_j > 0$  - assume a long position

if  $\alpha_j < 0$  - assume a short position

since N is large  $e_p \equiv 0$ , so ignore “unique” risks.

Remarks:

1.: In step 4 we could substitute independent (otherwise obtained) estimates of  $AR_i$ 's, and not use the historical averages.

2.: Notice that nowhere do we have to forecast future values of the factors.

3.: In forming the arbitrage portfolio we are implicitly assuming that the over and under pricing we believe exists will be eliminated in the future – to our advantage!

## Chapter 17

17.1. General equilibrium and uncertainty: Consider a two-date economy with two agents and one good. Assume two states of nature at the second date with probabilities  $\pi$  and  $1 - \pi$ .

The resources are the following:

	$t = 0$	$t = 1$	
		$\theta_1$	$\theta_2$
<i>Agent 1</i>	1	1	1
<i>Agent 2</i>	1	0	2

Agent 1's utility function is:  $U_1 = c_0^1 + \delta E(c^1(\theta))$

Agent 2's utility function is:  $U_2 = \ln c_0^2 + \delta E(\log c^2(\theta))$

The subjective discount rate is the same for both agents, (i.e.,  $\delta$ ).

- a. Comment on the form of the utility functions.
  - b. Determine the Pareto optimum.
  - c. Assuming markets are complete, compute the contingent prices, taking the good at date 0 as the numeraire.
  - d. Assume there is only one asset (a bond), which gives one unit of good in each state of nature.
    1. Give the price of the bond. *Hint*: Look at agent 1's program.
    2. Find the competitive equilibrium.
    3. Compute the equilibrium allocation for  $\pi = 0.5$  and  $\delta = \frac{1}{3}$ . Is this allocation Pareto optimal?
- 17.2. Summarize what we have learned about the Modigliani–Miller Theorem in the context of incomplete markets.
- 17.3. General equilibrium and uncertainty: Consider a two-period, two-agent economy with preferences and endowments as follows:

	<i>Endowments</i>		<i>Preferences</i>	
	$t = 0$	$t = 1$		
		$\theta = 1$	$\theta = 2$	
<i>Agent 1</i>	20	1	5	$U_1(\cdot) = c_0^1 + \pi(\theta_1)\frac{1}{2}\ln(c^1(\theta_1)) + \pi(\theta_2)\frac{1}{2}\ln(c^1(\theta_2))$ $U_2(\cdot) = c_0^2 + \pi(\theta_1)\frac{1}{2}c^2(\theta_1) + \pi(\theta_2)\frac{1}{2}c^2(\theta_2)$ (risk neutral)
<i>Agent 2</i>	30	15	15	

The state probabilities are:  $\pi(\theta_1) = \frac{1}{3}$ ,  $\pi(\theta_2) = \frac{1}{3}$ .

- a. Imagine that in this economy only the asset  $Q = (1,0)$  is traded. Construct the financial equilibrium. Will risk sharing take place? Is the initial allocation already Pareto optimal? How would you answer if the state probabilities were equal?
- b. Now we are in the initial setup where trade with  $(1,0)$  is still possible. Assume that asset  $R = (0, 1)$  is also traded. What will be the price of this asset? How much of the two assets will be exchanged? What is the post-trade allocation? Is it Pareto optimal?
- 17.4. Consider a two-date, two-agent economy, with the following initial endowments:

	$t = 0$	$t = 1$	
		$\theta = 1$	$\theta = 2$
<i>Agent 1</i>	4	6	1
<i>Agent 2</i>	6	2	4

The agents' utility functions are:

$$\text{Agent 1: } \frac{1}{2} \ln c_0^1 + E \ln c_\theta^1$$

$$\text{Agent 2: } \frac{1}{2} \ln c_0^2 + E \ln c_\theta^2$$

The probability of each state is  $\text{Prob}(\theta = 1) = 0.4$  and  $\text{Prob}(\theta = 2) = 0.6$ .

- a. Compute the initial utility of each agent.
- b. Suppose now there is a firm that generates an uncertain output at time  $t = 1$  of  $(2;3)$ . The firm owner is interested in consuming only at date 0, and accordingly he would like to sell his claim of ownership to the output of the firm. He solicits your advice as to the type of security that he should issue. Suppose first that he issues a stock, that is, a claim to two units of good in state  $\theta = 1$  and three units in state  $\theta = 2$ .
1. What is the price of the security/the stock market value of the firm?



2. What are the post-trade allocations?
3. What are the post-trade utilities?

Discuss intuitively how your results would have differed if the two states had been equally likely.

- c. What happens if, instead of a stock claim, the firm owner issues Arrow–Debreu securities. Discuss intuitively (no calculations needed) how your answer to these same questions would need to be modified.
- d. Now forget the firm. The situation is different: a foreign government would like to market a risk-free security, specifically a claim promising payment of two units of the good next period, regardless of the state. How much money will this bond issue generate? What are the post-trade allocation and utilities? Are you surprised?
- e. In one line, tell us how the problem would be altered if the bond issuer were not a foreign government but the local government.

- 17.5. Consider a two-period economy with two agents and one good. Assume two possible states of nature at the second period with same probability. The agents only care about their second-period consumption level. They maximize their expected utility.

Agent 1's utility function is :  $U_1 = c_1$

Agent 2's utility function is :  $U_2 = \log(c_2)$

- a. Agents can trade in a complete contingent security market system. Show, in full generality, that the price of the contingent security is the same for the two states of nature. Show that agent 2's consumption does not depend on the state of nature.
- b. Assume that, in addition to their endowments, the two agents each possess one-half of a firm. The firm invests two units of input in period 1. The output is available at period 2.

There are two possible technologies. With the first one, one unit of input produces one unit of good, independently of the state of nature. With the second one, one unit of input produces three units of good in the first state of nature and none in the other one.

Let  $x \in [0,1]$  be the part of input used with the second technology. The sale of the goods at the second period is the agents' only resource, so each of them receives  $(x + 1)$  in the first state of nature and  $(1 - x)$  in the second.

1. Assuming that the markets are complete. Write the agents' budget constraints as a function of the parameter  $x$ .
2. Determine the consumption and utility levels as a function of the parameter  $x$ . Show that the agents agree about the optimal choice of  $x$ . What value will they choose?
- c. From now on, suppose there are no markets for contingent security. However, the good can be traded on a spot market during the second period. Compute the equilibrium utility level attained by the agents as a function of  $x$ . Determine the preferred investment policy of each agent. Show that they disagree on the optimal level of  $x$ .
- d. Assume that the agents can trade two securities at period 1: one security pays one unit of good independently of the state of nature, and the other one is a share of the firm just described. Show that the markets are complete. What will agent 2 do in such a context?

(No formal proof is required.)

- 17.6 Consider two agents in the context of a pure exchange economy in which there are two dates ( $t = 0,1$ ) and two states at  $t = 1$ . The endowments of the two agents are different ( $e_1 \neq e_2$ ). Both agents have the same utility function :

$$U(c_0, c_1(\theta)) = \ln c_0 + E \ln c_1(\theta),$$

but they differ in their beliefs. In particular, agent 1 assigns probability  $\frac{3}{4}$  to state 1, while agent 2 assigns state 1 a probability  $\frac{1}{4}$ . The agents trade Arrow-Debreu claims and the supply of each claim is 1. Neither agent receives any endowment at  $t=1$ .

- a. Derive the equilibrium state claim prices. How are they related to the relative endowments of the agents ? How are the relative demands of each security related to the agents' subjective beliefs ?
- b. Suppose rather than trading state claims, each agent is given  $a_i$  units of a riskless security paying one unit in each future state. Their  $t=0$  endowments are otherwise unaffected. Will there be trade ? Can you think of circumstances where no trade will occur ?
- c. Now suppose that a risky asset is also introduced into this economy. What will be the effects ?
- d. Rather than introducing a risky asset, suppose an entrepreneur invents a technology that is able to convert  $x$  units of the riskless asset into  $x$  units each of  $(1,0)$  and  $(0,1)$ . How is  $x$  and the value of these newly created securities related ? Could the entrepreneur extract a payment for the technology ? What considerations would influence the magnitude of this payment ?

## Chapter 18

- 18.1. Assume the following speculator's program:

$$\max_f EU[\bar{c} + (p^f - p)f]$$

Is it true that  $f^* \begin{matrix} \geq \\ < \end{matrix} 0$  if and only if  $p^f \begin{matrix} \geq \\ < \end{matrix} Ep$  ?

- 18.2. Show that it is always better, that is, more profitable, for a producer to speculate on the futures market than on the physical market.

Assumptions:

- no uncertainty in production
- no basis risk

**DEFINITION** To speculate on the physical market means producing because an increase in price is expected, although in terms of the futures price, the production is not profitable (marginal cost is not covered).

- 18.3. The law of demand states that a price increase leads to a decrease in the quantity demanded. Comment on the applicability of the law of demand when the object being exchanged is a financial asset.
- 18.4. Provide at least one example where price and volume behavior is significantly different in a market with heterogeneously informed agents than it would be in a market where agents are homogeneously informed.
- 18.5. Consider a firm facing exchange rate risk for its output commodity: The production decision is made at date  $t$ , and the output is sold in foreign currency at date  $t + 1$ . Assume that no currency futures market exists; however, a market for a domestic financial asset is available. You can write the profit of the firm as follows:

$$\tilde{\pi} = py\tilde{e} - \frac{1}{2}y^2 + z(q^f - \tilde{q})$$

where

$p$  = the known foreign currency price

$y$  = the output

$\frac{1}{2}y^2$  = the cost function

$e$  = the exchange rate

$z$  = the number of shares of the domestic asset sold short at date  $t$

$\tilde{q}$  = the nominal payoff of the domestic financial asset at date  $t + 1$

$q^f$  = the date  $t$  price of the domestic financial asset

It is assumed that

$E(\tilde{q}) = q^f$  (i.e., the date  $t$  price of the financial asset is an unbiased predictor of the future price).

Suppose the firm maximizes a mean-variance utility function:

$$\max_{z,x} E(\tilde{\pi}) - \frac{\gamma}{2} \text{var}(\tilde{\pi})$$

- Write  $E(\tilde{\pi})$  and  $V(\tilde{\pi})$ .
- Compute and interpret the FOCs.
- Show that output is greater in the case of certainty than in the case of uncertainty.